Contingent Convertible Bond Pricing with a Black-Karasinski Credit Model

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Abstract

We extend the calculation by Turfus and Schubert (2016) of analytic pricing of CoCo bonds under a Hull-White short rate model for equity conversion intensity to a beta blend short rate model which encompasses both Hull-White (normal) and Black-Karasinski (lognormal) models. The asymptotic expansion assumes that the volatility of the conversion intensity is small. Results are calculated up to second order accuracy in the volatility. This is considered adequate for most practical purposes.

1 Introduction

A hybrid credit-equity model was proposed by Turfus and Schubert (2016) for the pricing of contingent convertible bonds (CoCos) with reduced form modelling of the occurrence of conversion events. The reader is referred to that paper and references therein for details about the defining characteristics of CoCo bonds and the various pricing approaches which have been proposed in the literature. The evolution of the equity conversion intensity was assumed to be governed by a diffusive process, and the price of the equity to which the bond converts by a jump-diffusion process, with a downward jump of a fixed relative amount k (< 0) occurring at the time τ of the conversion event. Thus the equity payoff was taken as $M(1+k)S_\tau^-$, with M the contractual conversion rate. The diffusive processes are further assumed to be (negatively) correlated.

The main achievement of the paper was to derive, by means of a perturbation expansion approach, approximate analytic solutions for the hybrid model under the assumption that the credit/conversion process was specifid by the (normal volatility) short rate model proposed by Hull and White (1990). Here we make two extensions to this work. Firstly, we embed the Hull-White model within a more general framework that includes in particular the (lognormal volatility) model of Black and Karasinski (1991). The latter has the advantage that it ensures intensities remain positive, which the Hull-White model does not. Secondly, we extend the first order approximation of Turfus and Schubert (2016) to include second order terms. These turn out, as we shall see, to be more important in the Black-Karasinski (lognormal volatility) case than they were for the Hull-White case.

We also note here that the model we propose here has many similarities than one of two models proposed by Cheridito and Xu (2015). They consider a structural model which seeks to represent the capital ratio of the CoCo-issuing bank directly, with conversion occurring when it drops below a trigger level. They also propose a reduced form model with many similarities to the one we propose here. Their models are solved by application of finite differences to the governing PDEs. Three significant differences between their reduced form model and the one we set out below are as follows:

- While we ensure positive intensities by using a Black-Karasinski model, they do so by using instead the square root model of Cox, Ingersoll and Ross (1985).
- While they take the payoff in the event of a conversion event at time τ to be $M(1+k)S_\tau^-$ unconditionally (in our notation), we suppose this amount to be capped, at least by the bond notional, on the basis

that it is highly improbably the capital ratios of a bank would trigger a conversion event when the equity conversion price exceeds the value of the convertible bond.

- Finally, Cheridito and Xu (2015) take default events explicitly into account insofar as they consider there to be a small probability that, when conversion is triggered, this does not save the bank from default, whence, rather than the stock falling to $(1+k)S_\tau^-$, it falls to zero.

The first of these is a residual difference which probably ought not to have too much impact on the conclusions reached since, as we shall see, the conversion intensity model only impacts through the integrated term variance and covariance of the intensity, which properties are likely to be similar under parallel calibrations of different models to the same market data. Our use of a possible cap makes our model more general than that of Cheridito and Xu (2015). We shall return to the final difference below but, suffice it to say, the incorporation of the impact of default events turns out to require only a trivial modification to our model. On this basis we suggest that all the most important features of the model of Cheridito and Xu (2015) are included in our model, with the advantage that analytic formulae are available to compute prices and risk sensitivities to a good degree of accuracy.

The structure of the paper is as follows. The founding assumptions of the mathematical model are set out in §2 below. The perturbation analysis leading to the calculation of the equity recovery value of the CoCo bond is set out in §3, the main results of which are Theorem 3.1 and Proposition 3.1. The accuracy of the asymptotic results and the importance of second order terms are considered briefly in §4. Consideration is given in §5 to how the impact of explicit default events can be taken into account in the manner of Cheridito and Xu (2015).

2 Mathematical Model

We consider an equity conversion process driven by a (stochastic) conversion intensity λ_t, the dynamics of which are governed by a beta blend short rate model. The method follows closely the similar calculation performed by Turfus and Schubert (2016) using a Hull-White short rate model. The equity price is taken to be given by a jump-diffusion process, with a downward jump of a fixed relative amount occurring at the time of a conversion event. The diffusive processes are potentially correlated. It having been observed by Turfus and Schubert (2016) that stochastic interest rates have little impact on pricing (at least in the non-callable case), we assume from the outset that rates are deterministic.

We shall find it convenient to work not with the conversion intensity λ_t directly, but with a reduced variable $\hat{y}_t$ satisfying the following canonical Ornstein-Uhlenbeck process:

$$d\hat{y}_t = -\alpha \hat{y}_t dt + \sigma_\lambda(t)\, dW_t^1 \tag{1}$$

where α is a positive constant specifying the mean reversion rate,[1] $\sigma_\lambda(t)$ is a bounded positive L^2 function and dW_t^1 is a Brownian motion for $t \in D_m := [t_0, \min\{\tau, T_m\}]$ where τ is the time of the first relevant CoCo bond conversion event and T_m is the longest maturity date for which we wish our model to be calibrated. This equation is well known to have a unique strong solution subject to these assumptions. The variable $\hat{y}_t$ is taken to be related to the conversion intensity λ_t by

$$(1-\beta)\,\lambda_t + \beta\overline{\lambda}(t) = \lambda^*(t)\exp\left(\frac{(1-\beta)\hat{y}_t}{|\overline{\lambda}(t)|^\beta}\right) \tag{2}$$

with $\beta \in [0,1)$ assumed and $\overline{\lambda}(t) > 0$ the instantaneous forward rate determined at the initial time t_0 with reference to market data and a suitable interpolation methodology. Here $\lambda^*(t)$ is chosen so as to satisfy the no-arbitrage condition that

$$E\left[e^{-\int_{t_0}^t \lambda_s\, ds}\right] = e^{-\int_{t_0}^t \overline{\lambda}(s)\, ds} \tag{3}$$

[1]It is a straightforward matter to loosen this assumption and specify instead that $\alpha(t)$ be a bounded positive L^1 function, whereupon the analysis below goes through effectively replacing $\alpha(v-u)$ throughout by $\int_u^v \alpha(s)ds$.

under the martingale (money market) measure for $t_0 < t \le T_m$. It is not difficult to show that Eq. (2) is consistent with Eq. (1) and furthermore that the local volatility of λ_t is

$$\frac{(1-\beta)\,\lambda_t + \beta\overline{\lambda}(t)}{|\overline{\lambda}(t)|^{\beta}}\sigma_\lambda(t).$$

This can be thought of as a displaced diffusion-type interpolation between the Hull-White (normal) and Black-Karasinski (lognormal) models. For the equity process we propose:

$$\frac{dS_t}{S_t} = (\overline{r}(t) - q(t) - k\lambda_t)\,dt + \sigma_S(t)\,dW_t^2 + kdn_t, \tag{4}$$

where S_t is the equity price, $q(t)$ its expected (continuous) dividend rate, $\overline{r}(t)$ the instantaneous forward rate of interest, dW_t^2 is a Brownian motion satisfying

$$\mathrm{corr}(W_t^1, W_t^2) = \rho_{\lambda S} \tag{5}$$

and n_t a Cox process with intensity λ_t giving rise to an equity price jump of size k with $-1 < k < 0$, contingent on an equity conversion event.

To proceed, we follow Turfus and Schubert (2016) and seek to price the survival-contingent flows separately from the conversion-contingent flows, expressing the PV of a CoCo bond with notional N as

$$V(t) = Nf_{\mathrm{surv}}(t) + Nf_{\mathrm{conv}}(t). \tag{6}$$

where $f_{\mathrm{surv}}(t)$ expresses the part of the PV generated by survival-contingent cash flows, i.e. the coupon payments and notional repayment, and $f_{\mathrm{conv}}(t)$ expresses the contribution to the PV resulting from cash flows contingent on a conversion event at time $\tau < T$, the bond maturity. We focus here on the calculation of the latter term and refer the reader to Turfus and Schubert (2016) for that of the former.

We propose as our model for the payoff of the CoCo bond upon conversion at $t = \tau$ that this is given by $\min\{MS_\tau^-(1+k), K\}$, with M the number of shares issued per unit notional on conversion, and $K \le 1$ specifying a cap on the value of the equity payoff.

Following Turfus (2016), we seek an approximate solution $f_{\mathrm{conv}}(t)$ under a "weak volatility" assumption. To this end we rescale both $\hat{y}_t$ and $\sigma_\lambda(t)$ by an asymptotic parameter ϵ defined by

$$\epsilon^2 := \frac{1}{\alpha(T_m - t_0)} \int_{t_0}^{T_m} \frac{\sigma_\lambda^2(t)}{|\overline{\lambda}(t)|^{2\beta}} dt$$

which we take to be small. Thus we define new scaled variables y_t and $\sigma_\lambda(t)$ by

$$y_t := \epsilon^{-1}\hat{y}_t, \tag{7}$$

$$\sigma_y(t) := \epsilon^{-1}\sigma_\lambda(t), \tag{8}$$

both taken to be $O(1)$ as $\epsilon \to 0$.

We further define an equity term variance

$$I_S(t_1, t_2) := \int_{t_1}^{t_2} \sigma_S^2(u)du, \tag{9}$$

a new characteristic stochastic equity price coordinate x_t such that

$$S_t = F(t)e^{x_t - \frac{1}{2}I_S(t_0,t)} \tag{10}$$

where

$$F(t) := S_{t_0}e^{\int_{t_0}^{t}(\overline{r}(s)-q(s)-k\overline{\lambda}(s))ds}, \tag{11}$$

a payoff function
$$M_0(x,t) := M(1+k)F(t)e^{x-\frac{1}{2}I_S(t_0,t)}$$
and a risky discount function
$$B(t_1,t_2) := e^{-\int_{t_1}^{t_2}(\overline{r}(s)+\overline{\lambda}(s))ds}. \tag{12}$$
We further suppose that the process $f_{\text{conv}}(t)$ can be expressed in terms of the new (stochastic) variables x_t and y_t as $h(x_t, y_t, t)$. We are interested in calculating $f_{\text{conv}}(t_0) = h(0,0,t_0)$.

Applying the well-known Feynman-Kac method to Eqs. (1), (4) and (5), we infer that the function $h(x,y,t)$ is governed by the following backward diffusion equation:
$$\mathcal{L}[h(x,y,t)] = -\left(\overline{\lambda}(t)+\Delta\lambda(y,t)\right)\min\{M_0(x,t),K\} + \Delta\lambda(y,t)\left(h + k\frac{\partial h}{\partial x}\right) \tag{13}$$
for $t \in D_m$ with final condition that $h(x,y,T)=0$, where $\mathcal{L}[.]$ is a standard forced diffusion operator given by
$$\mathcal{L}[.] := \frac{\partial}{\partial t} + \frac{1}{2}\left(\sigma_S^2(t)\frac{\partial^2}{\partial x^2} + 2\rho_{\lambda S}\sigma_S(t)\sigma_y(t)\frac{\partial^2}{\partial x\partial y} + \sigma_y^2(t)\frac{\partial^2}{\partial y^2}\right) - (\overline{r}(t)+\overline{\lambda}(t)) \tag{14}$$
and
$$\Delta\lambda(y,t) := \frac{1}{1-\beta}\left(\lambda^*(t)\exp\left(\frac{\epsilon(1-\beta)y}{|\overline{\lambda}(t)|^\beta}\right) - \overline{\lambda}(t)\right), \tag{15}$$
so that $\lambda_t \equiv \overline{\lambda}(t) + \Delta\lambda(y_t,t)$

3 Perturbation Analysis

In the absence of exact closed form solutions to Eq. (13), we seek a solution as a perturbation expansion in ϵ. To that end we write
$$\frac{\lambda^*(t)-\overline{\lambda}(t)}{1-\beta} = \epsilon\lambda_1^*(t) + \epsilon^2\lambda_2^*(t) + \epsilon^3\lambda_3^*(t) + O(\epsilon^4)$$
where we know from Turfus (2016) that the no-arbitrage condition Eq. (3) requires us to choose $\lambda_1^*(t) = \lambda_3^*(t) = 0$ and
$$\lambda_2^*(t) = \frac{\overline{\lambda}(t)}{|\overline{\lambda}(t)|^\beta}\int_{t_0}^t e^{-\alpha(t-u)}I_y(t_0,u)\frac{\overline{\lambda}(u)}{|\overline{\lambda}(u)|^\beta}du - \tfrac{1}{2}(1-\beta)\frac{\overline{\lambda}(t)}{|\overline{\lambda}(t)|^{2\beta}}I_y(t_0,t). \tag{16}$$
with
$$I_y(t_1,t_2) := \int_{t_1}^{t_2} e^{-2\alpha(t_2-u)}\sigma_y^2(u)du. \tag{17}$$
By expanding the exponential in Eq. (15) as a power series about $\epsilon = 0$ and gathering together like powers, we can also write
$$\Delta\lambda(y,t) = \epsilon\lambda_1(y,t) + \epsilon^2\lambda_2(y,t) + O(\epsilon^3) \tag{18}$$
where
$$\lambda_1(y,t) = \frac{\overline{\lambda}(t)}{|\overline{\lambda}(t)|^\beta}y, \tag{19}$$
$$\lambda_2(y,t) = \lambda_2^*(t) + \tfrac{1}{2}(1-\beta)\frac{\overline{\lambda}(t)}{|\overline{\lambda}(t)|^{2\beta}}y^2. \tag{20}$$

Note that by construction we have $\lambda_i^*(t) = \lambda_i(0,t)$, $i = 1,2,\ldots$

To solve Eq. (13) in it asymptotic representation we pose a perturbation expansion

$$h(x,y,t) = h_0(x,t) + \epsilon h_1(x,y,t) + \epsilon^2 h_2(x,y,t) + O(\epsilon^3). \tag{21}$$

Substituting the above expansions into Eq. (13), solving for successive power of ϵ, setting $t = t_0$ and reverting to unscaled notation, we obtain the following first order accurate result:

Theorem 3.1 *The value of the equity recovery payoff on a CoCo bond can be estimated under our modelling assumptions as follows:*

$$f_{conv}(t_0) = \int_{t_0}^{T} B(t_0,v)\overline{\lambda}(v)\Big(M(1+k)F(v)\mathcal{M}[N(-d_1(x,t_0,v))]\Big|_{x=0} + KN(d_2(0,t_0,v))\Big)\,dv + O(\epsilon^2) \tag{22}$$

where the operator $\mathcal{M}$ is defined by

$$\mathcal{M}[f(x)] := \left(1 + \frac{I_R(t_0,v)}{|\overline{\lambda}(v)|^{\beta}} - \int_{t_0}^{v} I_R(t_0,u)\frac{\overline{\lambda}(u)}{|\overline{\lambda}(u)|^{\beta}}du\left(1+k+k\frac{\partial}{\partial x}\right)\right) f(x) \tag{23}$$

and

$$I_R(t_1,t_2) := \rho_{\lambda S}\int_{t_1}^{t_2} e^{-a(t_2-u)}\sigma_\lambda(u)\sigma_S(u)du, \tag{24}$$

$$d_2(x,t,v) := \frac{\ln M_0(x,v) - \ln K}{\sqrt{I_S(t,v)}}, \tag{25}$$

$$d_1(x,t,v) := d_2(x,t,v) + \sqrt{I_S(t,v)}, \tag{26}$$

$$N(x) := \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x} \exp\left(-\tfrac{1}{2}u^2\right)du. \tag{27}$$

Proof. For the proof of Theorem 3.1, see Appendix A. □

We make the observation that the form of Eq. (22) is akin to a CVA calculation, giving the PV of a payment at default of an equity underlying capped at a collateral level K. Ignoring the second two terms on the r.h.s. of Eq. (23) and setting $\mathcal{M} \equiv 1$ gives the value ignoring wrong-way risk and jump risk. The neglected terms correct for the latter effects. It is worth observing that, although the wrong-way risk is initially (for short times to maturity) driven primarily from the second term in direct proportion to the covariance $I_R(.)$, as time to maturity is increased and/or the default/conversion intensity increases, the impact of the covariance is diminished by the impact of the remaining third term.

Although the result of Theorem 3.1 gives the equity recovery value to first order in ϵ, it is at times necessary to use a more accurate approximation, particularly for the Black-Karasinski case where $\sigma_\lambda(t)$ will not in general be particularly small. This can become an issue when the time to maturity of the bond exceeds about 10 years. Errors can exceed about 50 bp of notional rendering the first order approximation less than adequate (see §4 below). To that end, the process described in Appendix A can be extended to allow second order terms to be calculated. The following result is obtained.

Proposition 3.1 *A second order accurate estimate for $f_{conv}(t_0)$ is obtained by adding to the contribution of Eq. (22) the following adjustment:*

$$\Delta f_{conv}(t_0) := \int_{t_0}^{T} B(t_0,v)\overline{\lambda}(v)M(1+k)F(v)\psi(0,t_0,v)\,dv \tag{28}$$

Details of the derivation of this expression and definition of $\psi(.)$ are provided in Appendix B. As can be seen from the form of $\psi(.)$, a quadratic dependence on the correlation level $\rho_{\lambda S}$ is introduced at this order of approximation through $I_R(.)$, and an explicit (quadratic) dependence on the conversion intensity volatility level, through $I_\lambda(.)$. Note that the contribution from the latter vanishes when the jump parameter $k \to 0$.

We observe that the Hull-White case ($\beta = 1$) was excluded under our initial assumptions as a singular limit of Eq. (2). However it will be observed that β can conveniently be set to unity in the above expressions for $f_{\mathrm{conv}}(t_0)$ and $\Delta f_{\mathrm{conv}}(t_0)$. It can be confirmed that the expressions deduced are consistent with those obtained by Turfus and Schubert (2016), leading us to conclude that the results presented herein are indeed valid for all $\beta \in [0, 1]$.

4 Results and comparative analysis

Computations were made for the important Black-Karasinski case of the equity recovery value of a range of CoCo bonds under various market assumptions based on the zeroth, first and second order expansions. We chose values of $\lambda = 6\%$, $\sigma_S = 30\%$, $\sigma_\lambda = 50\%$, $a = 35\%$, $\rho_{\lambda S} = -0.6$ and $k = -0.8$. The notional N was taken to be 100; the initial share price S_{t_0}, the share issue rate M and the cap K were all taken to be 1. The results are compared in Figs. 1 and 2, with the time to maturity in Fig. 2 taken to be 15 years.

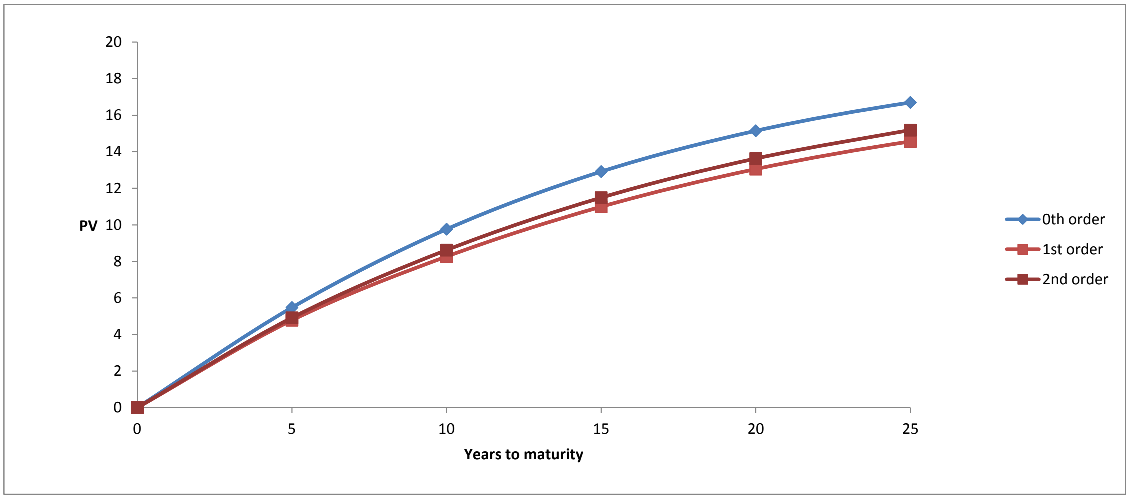

Figure 1: Asymptotic estimates of equity recovery value for various CoCo bond maturity dates.

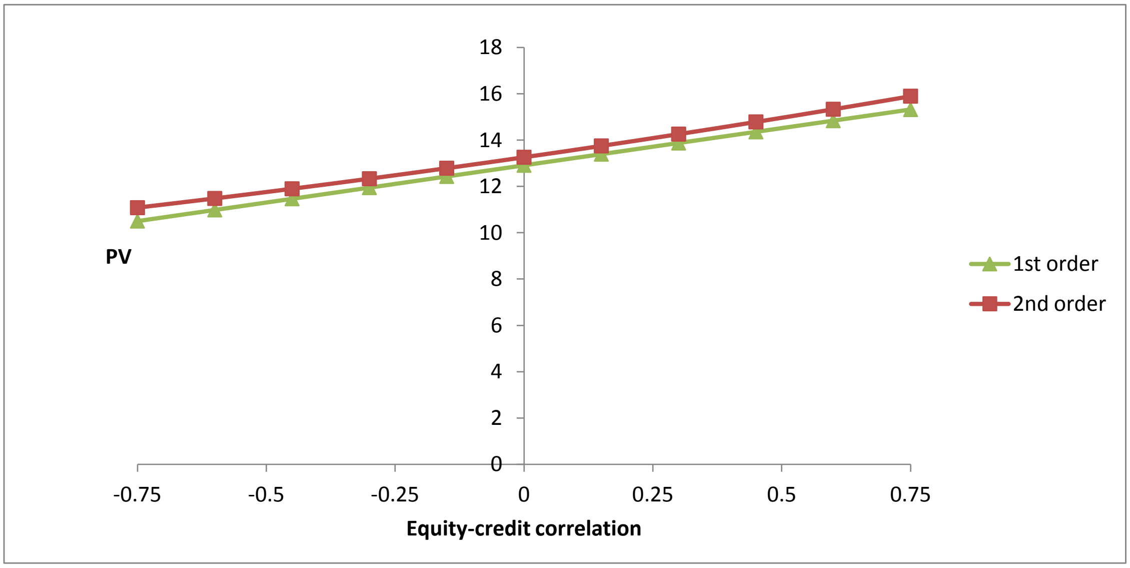

Figure 2: Asymptotic estimates of equity recovery value with various assumed conversion intensity-equity correlation levels.

It is observed that, while the second order corrections for maturities of 10 years or below were less than 20 bp of notional, this quickly increased to 50-60 bp for maturities of 15y or more for typical market circumstances. This is larger by a factor of about two than the size of the errors observed by Turfus and Schubert (2016) for an assumed *normal* conversion intensity volatility. It is therefore suggested that, for longer maturity bonds, the second order expansion should be used in preference to the first order. However, given the large decrease in size of the second order correction compared to the first order, there is good reason to believe it justifiable to ignore the contribution due to third and higher order terms and to suppose our second order expansion to be adequate for most practical purposes.

5 Extensions of the Main Result

Cheridito and Xu (2015) in their reduced form model propose an additional feature not considered in the above model, namely the possibility of a default being triggered by the conversion event. They take the probability of this happening (with zero resultant conversion payment) to be α ($\ll 1$), whence the probability that a payoff of $M(1+k)S_\tau^-$ is received is $1-\alpha$. But this feature is easily incorporated into our framework: instead of a deterministic jump of k, we face a binary choice between a jump of k and a jump of -1. Consequently, rather then the (uncapped) payoff being $M(1+k)S_\tau^-$, we must consider it to be effectively given by $(1-\alpha)M(1+k)S_\tau^-$. Having made this change, to ensure that the cap is applied correctly to the effective payoff, we should set it to $(1-\alpha)K$ rather than K. Finally, rather than the jump-compensating drift of $-k\lambda_t$ in Eq. 4, we need to use a drift of $-((1-\alpha)\times k+\alpha\times -1)\lambda_t = -(k-\alpha(1+k))\lambda_t$. In conclusion our model only needs to be modified by changing

$$\begin{aligned} k &\to k-\alpha(1+k) \\ K &\to (1-\alpha)K \end{aligned} \tag{29}$$

to incorporate the impact of default at conversion with probability α. With this change, all our main results can be used as presented.

We add in passing that it was observed by Turfus and Schubert (2016) that it was necessary to choose rather higher values of k than might have been expected *a priori* to achieve a consistent calibration of the model to market prices of CoCo bonds. This was interpreted by suggesting that the large jump apparently

being priced in was not just from the impact of the conversion event itself but from other factors impacting the stock price at around the same time as the conversion event. The occurrence of a default immediately in the wake of a conversion event is a good example of such. On this basis we propose that factoring in a small probability of "default at conversion" is probably an appropriate measure to take in looking to calibrate the above model.

We make the further observation that Cheridito and Xu (2015) propose that, if there is no immediate default upon conversion, there remains a risk of a subsequent default, with the relevant intensity given by $\mu_t = \beta\lambda_t$ for some constant $0 < \beta < 1$. The value of this constant does not impact on the pricing but is used by the authors in their calibration process, insofar as they look to have the model price CDS rates in a manner consistent with this double-event model. By assuming α and β to be known *a priori*, they are able to use the CDS calibration process to infer the (otherwise) unknown conversion intensity λ_t. While this is superficially attractive, it is arguably only exchanging one intractable problem for another. For, if we suppose the conversion intensity to be intrinsically unknown, we are not in a position to infer its relationship to the default intensity, other than that it will be greater. The approach of Cheridito and Xu (2015) is essentially to suppose the following:

$$\mu_t = \begin{cases} \alpha\lambda_t, & t < \tau, \\ \beta\lambda_t, & t > \tau, \end{cases}$$

in other words a piecewise linear relationship. While this ansatz would appear to be capable of yielding information about λ_t from the better understood μ_t, confidence in any conclusions drawn is only merited to the degree that α and β are known. Of course, under the approach of Cheridito and Xu (2015), the assignment of both values is necessarily an *ad hoc* process. In the end, many more equally plausible calibration approaches could be used, such as inferring μ_t directly from CDS and postulating that $\lambda_t = \mu_t + s(t)$, for some deterministic function $s(t)$, chosen to attain consistency with CoCo market prices.

A Proof of Theorem 3.1

We derive Eq. (22), solving Eq. (13) in the standard manner by successive levels of approximation. At zeroth order we must solve

$$\mathcal{L}[h_0(x,t)] = -\overline{\lambda}(t)\min\{M_0(x,t), K\}$$

for $t \in D_m$ with final condition that $h_0(x,T) = 0$. This can be achieved by means of the following readily obtainable Green's function for the canonical diffusion operator $\mathcal{L}[.]$:

$$G(x,y,t;\xi,\eta,v) = B(t,v)H(v-t)\phi(x-\xi, ye^{-\alpha(v-t)} - \eta; R(t,v)) \tag{30}$$

where $B(t,v)$ is the risky discount factor defined in Eq. (12) above, $H(.)$ is the Heaviside step function and $\phi(x_1,x_2;R(t,v))$ is a bivariate gaussian probability distribution function with mean $\mathbf{0}$ and covariance matrix

$$R(t,v) = \begin{pmatrix} I_S(t,v) & I_\rho(t,v) \\ I_\rho(t,v) & I_y(t,v) \end{pmatrix} \tag{31}$$

where

$$I_\rho(t_1,t_2) = \rho_{\lambda S}\int_{t_1}^{t_2} e^{-a(t_2-u)}\sigma_y(u)\sigma_S(u)du,$$

and $I_S(t_1,t_2)$ and $I_y(t_1,t_2)$ are defined in Eqs. (9) and (17), respectively. Making use of Eq. (30) we obtain

$$\begin{aligned} h_0(x,t) &= \int_t^T \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} G(x,y,t;\xi,\eta,v)\overline{\lambda}(v)\min\{M_0(\xi,v),K\}\,d\xi\,d\eta\,dv \\ &= \int_t^T \overline{\lambda}(v)B(t,v)\Big(e^{x-\frac{1}{2}I_S(t_0,t)}M(1+k)F(v)N(-d_1(x,t,v)) + KN(d_2(x,t,v))\Big)dv \end{aligned} \tag{32}$$

where $d_1(.)$, $d_2(.)$ and $N(.)$ are defined in Eqs. (25)-(27) above. Continuing to first order we find

$$\mathcal{L}[h_1(x,y,t)] = \lambda_1(y,t)\left(-\min\{M_0(x,t),K\} + h_0 + k\frac{\partial h_0}{\partial x}\right)$$

for $t \in D_m$ with final condition that $h_1(x,y,T) = 0$. The solution can be written

$$h_1(x,y,t) = h_{1,0}(x,t) + y h_{1,1}(x,t),$$

where, applying our Green's function as previously and integrating over η, we find

$$\begin{aligned}
h_{1,0}(x,t) &= \frac{\partial}{\partial x}\int_t^T\int_{-\infty}^{\infty} B(t,v)N'\left(\frac{\xi - x}{\sqrt{I_S(t,v)}}\right) I_\rho(t,v)\frac{\overline{\lambda}(v)}{|\overline{\lambda}(v)|^\beta} \\
&\qquad \left(\min\{M_0(\xi,v),K\} - \left(1 + k\frac{\partial}{\partial \xi}\right) h_0(\xi,v)\right) d\xi\, dv \\
&= e^{x-\frac{1}{2}I_S(t_0,t)} M(1+k)\int_t^T F(v)B(t,v)I_\rho(t,v)\frac{\overline{\lambda}(v)}{|\overline{\lambda}(v)|^\beta} N(-d_1(x,t,v))\, dv \\
&\quad - e^{x-\frac{1}{2}I_S(t_0,t)} M(1+k)\int_t^T F(v)B(t,v)\overline{\lambda}(v) \\
&\qquad \int_t^v I_\rho(t,u)\frac{\overline{\lambda}(u)}{|\overline{\lambda}(u)|^\beta}\left(1 + k + k\frac{\partial}{\partial x}\right) N(-d_1(x + I_S(t,u),t,v))\, du\, dv. \qquad (33)
\end{aligned}$$

Likewise we find

$$\begin{aligned}
h_{1,1}(x,t) &= \int_t^T\int_{-\infty}^{\infty} B(t,v)N'\left(\frac{\xi - x}{\sqrt{I_S(t,v)}}\right) e^{-\alpha(v-t)}\frac{\overline{\lambda}(v)}{|\overline{\lambda}(v)|^\beta} \\
&\qquad \left(\min\{M_0(\xi,v),K\} - \left(1 + k\frac{\partial}{\partial \xi}\right) h_0(\xi,v)\right) d\xi\, dv \\
&= \int_t^T B(t,v)e^{-\alpha(v-t)}\frac{\overline{\lambda}(v)}{|\overline{\lambda}(v)|^\beta}\left(e^{x-\frac{1}{2}I_S(t_0,t)} M(1+k)F(v)N(-d_1(x,t,v)) + KN(d_2(x,t,v))\right) dv \\
&\quad - \int_t^T B(t,v)\overline{\lambda}(v)\int_t^v e^{-\alpha(u-t)}\frac{\overline{\lambda}(u)}{|\overline{\lambda}(u)|^\beta}\Big(e^{x-\frac{1}{2}I_S(t_0,t)} M(1+k)^2F(v)N(-d_1(x + I_S(t,u),t,v)) \\
&\qquad + KN(d_2(x,t,v))\Big) du\, dv. \qquad (34)
\end{aligned}$$

Substituting the above expressions into Eq. (21), setting $t = t_0$ and reverting to unscaled notation, we obtain Eq. (22).

This concludes the proof of the theorem. □

B Derivation of Second Order Terms

For notational convenience we define at this stage the following differential operators:

$$\begin{aligned}
D_0 &:= \frac{\partial}{\partial x}, \\
D_1 &:= 1 + \frac{\partial}{\partial x}, \\
D_2 &:= 1 + k\frac{\partial}{\partial x}, \\
D_3 &:= 1 + k + k\frac{\partial}{\partial x}.
\end{aligned}$$

and an unscaled equity conversion variance

$$I_\lambda(t_1,t_2) := \epsilon^2 I_y(t_1,t_2).$$

Continuing in the same vein as Appendix A we find that the second order contribution $h_2(x,y,t)$ satisfies the following:

$$\mathcal{L}[h_2(x,y,t)] = \lambda_2(y,t)\left(-\min\{M_0(x,t),K\} + D_2 h_0(x,t)\right) + \lambda_1(y,t)\Big(D_2 h_{1,0}(x,t) + yD_2 h_{1,1}(x,t)\Big), \quad (35)$$

with final condition $h_2(x,y,T)=0$. From the form of Eq. (35), we are led to suppose a solution of the form

$$h_2(x,y,t) = h_{2,0}(x,t) + yh_{2,1}(x,t) + y^2 h_{2,2}(x,t), \quad (36)$$

Applying our Green's function Eq. 30 in the standard way, carrying out the required integrations and simplifying, we find in particular that

$$\epsilon^2 h_{2,0}(x,t) = e^{x-\frac{1}{2}I_S(t_0,t)} M(1+k)\int_t^T F(v)B(t,v)\overline{\lambda}(v)\psi(x,t,v)dv \quad (37)$$

where, reverting for convenience to unscaled notation, we have

$$\begin{aligned}
\psi(x,t,v) := {} & \tfrac{1}{2}(1-\beta)\frac{I_R^2(t,v)}{|\overline{\lambda}(v)|^{2\beta}}N(-d_1(x,t,v)) - \frac{I_R(t,v)}{|\overline{\lambda}(v)|^{\beta}}\int_t^v \frac{\overline{\lambda}(u)}{|\overline{\lambda}(u)|^{\beta}} I_R(t,u)\,du\,D_3 N(-d_1(x,t,v)) \\
& - \int_t^v e^{-a(v-u)}\frac{\overline{\lambda}(u)}{|\overline{\lambda}(u)|^{2\beta}}\Big(kI_\lambda(t,u) + I_R^2(t,u)\left((1+k)D_1^2 - D_0^2\right)\Big)N(-d_1(x,t,v)), \\
& - \tfrac{1}{2}(1-\beta)\int_t^v \frac{\overline{\lambda}(u)}{|\overline{\lambda}(u)|^{2\beta}}(1+k)I_R^2(t,u)\,du\,(D_1^2 - D_0^2)N(-d_1(x,t,v)) \\
& + \int_t^v \frac{\overline{\lambda}(u)}{|\overline{\lambda}(u)|^{\beta}} I_R(t,u)\int_t^u \frac{\overline{\lambda}(w)}{|\overline{\lambda}(w)|^{\beta}} I_R(t,w)\,dw\,du\,D_3^2 D_1 N(-d_1(x,t,v)) \\
& + (1+k)\int_t^v \frac{\overline{\lambda}(u)}{|\overline{\lambda}(u)|^{\beta}}\int_t^u e^{-a(u-w)}\frac{\overline{\lambda}(w)}{|\overline{\lambda}(w)|^{\beta}} \\
& \qquad \Big(kI_\lambda(t,w)D_3 + I_R^2(t,w)\left((1+k)D_3D_1^2 - D_2D_0^2\right)\Big)N(-d_1(x,t,v))\,dw\,du. \quad (38)
\end{aligned}$$

The final $O(\epsilon^2)$ contribution to $f_{\mathrm{conv}}(t_0)$ is then given by $\epsilon^2 h_2(0,0,t_0)$. Clearly the y-dependent terms in Eq. (36) do not contribute (whence we do not produce them here). Substituting in from Eqs. (37)-(38) we obtain the result in Eq. (28).

References

Black, F., P. Karasinski (1991) 'Bond and Option Pricing when Short Rates are Lognormal' Financial Analysts Journal, Vol. 47(4), pp. 52–59.

Cheridito, P., Z. Xu (2015) 'Pricing and Hedging CoCos' `https://papers.ssrn.com/sol3/Papers.cfm?abstract_id=2201364`

Cox, J.C., J.E. Ingersoll and S.A. Ross (1985) 'A Theory of the Term Structure of Interest Rates', Econometrica Vol. 53, pp. 385–407.

Hull, J., A. White (1990:) 'Pricing Interest Rate Derivative Securities', The Review of Financial Studies, Vol. 3, pp. 573–592.

Turfus, C. (2016) 'Caplet Pricing in the Beta Blend Short Rate Model' (in preparation).

Turfus, C., A. Shubert (2016) 'Analytic Pricing of CoCo Bonds' `https://archive.org/details/AnalyticPricingCoCoBonds`.